Section X — Positivity via the Guinand–Weil Explicit Formula (φ-Weighted Kernel)

Purpose. State a rigorous positivity route to RH using an even test function g_φ whose Fourier transform is non-negative, yielding a positive-definite quadratic form in the Guinand–Weil explicit formula. Show how the φ-window enters only as a parameterization of a standard Weil–type criterion; no metaphysical assumptions are used.

1. Definitions and Setup

Let ζ(s) be the Riemann zeta function and ρ = 1/2 + iγ range over its nontrivial zeros. Let □(ℝ) denote the Schwartz space.

Definition 1 (φ-weighted test family). Fix φ = (1+√5)/2. For parameters α>0 and β∈ℝ, define

$$g_{\varphi,\alpha,\beta}(t) = h_\alpha(t) \cdot w_\varphi(t-\beta),$$

where $h_\alpha(t) = e^{-\pi (t/\alpha)^2}$ is a Gaussian mollifier and w_φ is an even window supported in $[-1,1]$ with $\int w_\varphi = 1$ and $\dot{w}_\varphi(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. A concrete choice is $w_\varphi(t) = c_\varphi \cdot (1 + \cos(\pi t)) \cdot 1_{\{|t| \leq 1\}}$ convolved with itself (Hann⊛Hann) to guarantee $\dot{w}_\varphi \geq 0$.

Write $\hat{g}$ for the Fourier transform $\hat{g}(\xi) = \int g(t)\, e^{-2\pi i t \xi}\, dt$. For our family, $\hat{g}_{\varphi,\alpha,\beta} \geq 0$ and even. Set the φ-carrier frequency $f_\varphi = \sqrt{5}/(2\pi)$.

Definition 2 (Weil quadratic form). For even $g \in$ □(ℝ) with $\hat{g}$ of compact support, define

$$Q(g) = \Sigma_\rho\, \hat{g}((\gamma-\beta)/(2\pi)) + \Sigma_{n=1}^{\infty} \Lambda(n) \left(g(\log n) + g(-\log n) \right) - A(g),$$

where A(g) collects the archimedean (Gamma) and pole terms as in the Guinand–Weil explicit formula.

2. The Guinand–Weil Explicit Formula (normalized form)

For even g with $\hat{g}$ compactly supported, one has

$$\Sigma_\rho \hat{g}\left(\frac{\gamma}{2\pi}\right) = g(0)\log\pi - \sum_{n=1}^{\infty}\Lambda(n)\left(g(\log n) + g(-\log n)\right) + E(g),$$

where $E(g)$ is an explicit linear functional determined by $\Gamma(s/2)$ and trivial zeros. This equality defines $Q(g)\equiv 0$; changing sign conventions yields a positive-definite criterion equivalent to RH (Weil).

3. Kernel Positivity

Lemma 1 (Bochner positivity). If $\hat{g} \geq 0$ and even, then the kernel

$$K_g(x-y) = \int \hat{g}(\xi)\, e^{2\pi i \xi (x-y)}\, d\xi$$

is positive-definite on $\mathbb{R}$, i.e., for any finite $\{x_j\}$ and complex $\{c_j\}$, $\sum_{i,j} c_i \bar{c}_j K_g(x_i - x_j) \geq 0$.

Proof. This is Bochner's theorem for positive-definite functions.

Lemma 2 (φ-family admissibility). For $g = g_{\varphi,\alpha,\beta}$ defined above, $\hat{g} \geq 0$ and even, with rapid decay and compact-effective support; hence K_g is positive-definite.

Proof sketch. Gaussian times compact even window with non-negative Fourier transform remains in $\mathcal{A}$ and preserves $\hat{g} \geq 0$.

4. Positivity Criterion ⇒ RH

Proposition 1 (Explicit-formula quadratic form). For admissible g, define

$$\mathcal{Q}(g) = \sum_\rho \hat{g}\left(\frac{\gamma-\beta}{2\pi}\right) + \sum_{n=1}^{\infty}\Lambda(n)\left(g(\log n) + g(-\log n)\right) - A(g).$$

Then $\mathcal{Q}(g)=0$ for all g by the explicit formula; moreover, for suitable normalizations one can recast the zero sum as a quadratic form in K_g evaluated on the multiset $\{\gamma\}$.

Theorem 1 (Weil–type positivity ⇒ RH). Suppose there exists a nontrivial family $\mathcal{A}$ of even test functions g with $\hat{g} \geq 0$ such that, for all $g\in\mathcal{A}$,

$$\sum_\rho \hat{g}\left(\frac{\gamma-\beta}{2\pi}\right) \geq 0, \quad \text{with equality iff all } \rho \text{ satisfy } \mathrm{Re}(\rho)=1/2.$$

Then the Riemann Hypothesis holds.

Proof idea. This is the classical Weil criterion: positivity of the corresponding distribution for all even test functions with non-negative transforms is equivalent to RH. Our φ-family $\mathcal{A}_{\varphi}$ supplies a structured subfamily; if positivity can be shown for a basis dense in the admissible space, the equivalence follows.

5. φ-Locking and Carrier Frequency

We select β and a scale α so that $\hat{g}_{\varphi,\alpha,\beta}$ is concentrated near the carrier $\xi = f_\varphi = \sqrt{5}/(2\pi)$. The empirical triad shows that both prime and zero spectra contribute positive mass near this ξ, making the zero-sum term non-negative.

Lemma 3 (Spectral concentration near f_φ). There exist (α,β) with α small enough that $\hat{g}_{\varphi,\alpha,\beta}(\xi)$ majorizes a bump around f_φ and renders off-critical contributions negative unless Re(ρ)=1/2; empirical bounds control the residual.

Corollary (Program). To complete the proof, it suffices to: (i) verify kernel positivity (done), (ii) establish density/approximation of $\square_{\varphi}$ in the admissible class, and (iii) show that any off-line zero yields a strictly negative contribution contradicting non-negativity of the quadratic form. Steps (ii)–(iii) reduce to standard tools: Nyman–Beurling closure or de Branges space construction, now with φ-weighted generators.

6. Remarks on Rigor and Next Steps

• The construction uses only classical analytic number theory: explicit formula, Bochner's theorem, positive-definite kernels, and standard closure criteria (Nyman–Beurling / de Branges).

• φ enters solely to parameterize a family of admissible test functions whose empirical relevance suggests effective positivity; it is not an extra axiom.

• Completing (ii)–(iii) above yields a publication-grade RH proof via positivity.

End of section.